

A Simple Generalization of a Result for Random Matrices with Independent Sub-Gaussian Rows

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Abstract—In this short note, we give a very simple but useful generalization of a result of Vershynin (Theorem 5.39 of [1]) for a random matrix with independent sub-Gaussian rows. We also explain with an example where our generalization is useful.

I. INTRODUCTION

In this note, we obtain a generalization of a result of Vershynin, Theorem 5.39 of [1]. This result bounds the minimum and maximum singular values of an $N \times n$ matrix \mathbf{W} with mutually independent, sub-Gaussian, and isotropic rows. We use $\|\cdot\|$ to denote the l_2 norm of a vector or the induced l_2 norm of a matrix, and we use $'$ to denote matrix or vector transpose. Let $\mathbf{W} = [\mathbf{w}_1, \mathbf{w}_2, \dots, \mathbf{w}_N]'$. Thus, \mathbf{w}_j is its j -th row. As explained in [1, Section 5.2], “isotropic” means that $\mathbb{E}[\mathbf{w}_j \mathbf{w}_j'] = \mathbf{I}$ where \mathbf{I} is the identity matrix. In Remark 5.40 of [1], this result is generalized to the case where the rows \mathbf{w}_j are not isotropic but have the same second moment matrix, $\mathbb{E}[\mathbf{w}_j \mathbf{w}_j']$ for all the N rows. As explained in [1], a sub-Gaussian random variable (r.v.), x , is one for which the following holds: there exists a constant K_g such that $\mathbb{E}[|x|^p]^{1/p} \leq K_g \sqrt{p}$ for all integers $p \geq 1$. The smallest such K_g is referred to as the sub-Gaussian norm of x , denoted $\|x\|_{\varphi_2}$. Thus, $\|x\|_{\varphi_2} = \sup_{p \geq 1} p^{-1/2} \mathbb{E}[|x|^p]^{1/p}$. A sub-Gaussian random vector, \mathbf{x} , is one for which, for all unit norm vectors \mathbf{z} , $\mathbf{x}'\mathbf{z}$ is sub-Gaussian. Also, its sub-Gaussian norm, $\|\mathbf{x}\|_{\varphi_2} = \sup_{\|\mathbf{z}\|=1} \|\mathbf{x}'\mathbf{z}\|_{\varphi_2}$.

Let K denote the maximum of the sub-Gaussian norms of the rows of \mathbf{W} . Theorem 5.39 of [1] shows that, for any $t > 0$, with probability at least $1 - \exp(-c_K t^2)$, the minimum singular value of \mathbf{W} is more than $\sqrt{N} - (C_K \sqrt{n} + t)$ and the maximum is less than $\sqrt{N} + (C_K \sqrt{n} + t)$. Here C_K and c_K are numerical constants that depend only on K . These bounds are obtained by bounding the deviation of $\frac{1}{N} \mathbf{W}'\mathbf{W}$ from its expected value, which is equal to \mathbf{I} .

Our generalization of this result does two extra things. First, it bounds $\|\mathbf{W}'\mathbf{W} - \mathbb{E}[\mathbf{W}'\mathbf{W}]\|$, even when the different rows of \mathbf{W} do not have the same second moment matrix. Second, it states a separate result that bounds $\|\mathbf{W}\mathbf{z}\|^2$ for one specific vector \mathbf{z} . This bound clearly holds with much higher probability than the bound on $\|\mathbf{W}'\mathbf{W} - \mathbb{E}[\mathbf{W}'\mathbf{W}]\|$. The proof approach for getting our result is the same as that used to get Theorem 5.39 of [1]. Thus, our generalization would be obvious to a reader who understands the proof of that result. However, it is a useful addition to the literature for readers who would like to just use results from [1] in their work, without having to understand all their proof techniques.

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II. OUR RESULT

Theorem 2.1. Suppose that \mathbf{w}_j , $j = 1, 2, \dots, N$, are n -length independent, sub-Gaussian random vectors with sub-Gaussian norms bounded by K . Let

$$\mathbf{D} := \frac{1}{N} \sum_{j=1}^N \mathbf{w}_j \mathbf{w}_j' - \frac{1}{N} \sum_{j=1}^N \mathbb{E}[\mathbf{w}_j \mathbf{w}_j'].$$

For an $\varepsilon > 0$,

- 1) for a given vector \mathbf{z} , with probability (w.p.) at least $1 - 2 \exp(-c \min(\varepsilon, \varepsilon^2)N)$,

$$|\mathbf{z}'\mathbf{D}\mathbf{z}| \leq 4\varepsilon K^2 \|\mathbf{z}\|^2;$$

- 2) w.p. at least $1 - 2 \exp(n \log 9 - c \min(\varepsilon, \varepsilon^2)N)$,

$$\|\mathbf{D}\| \leq 4\varepsilon K^2.$$

for a numerical constant c .

Here, and throughout the paper, the letter c is reused to denote different numerical constants.

Proof: The proof is given in the Supplementary Material. It follows using the approach developed in [1]. ■

Remark 2.2. Recall that $\mathbf{W} := [\mathbf{w}_1, \mathbf{w}_2, \dots, \mathbf{w}_N]'$ and so $\sum_j \mathbf{w}_j \mathbf{w}_j' = \mathbf{W}'\mathbf{W}$. Thus the first claim implies that $\frac{1}{N} \|\mathbf{W}\mathbf{z}\|^2 = \frac{1}{N} \mathbf{z}'\mathbf{W}'\mathbf{W}\mathbf{z}$ lies in the interval $(\frac{1}{N} \mathbf{z}'\mathbb{E}[\mathbf{W}'\mathbf{W}]\mathbf{z}) \pm 4\varepsilon K^2 \|\mathbf{z}\|^2$ w.p. $\geq 1 - 2 \exp(-c \min(\varepsilon, \varepsilon^2)N)$. Using Weyl's inequality, the second claim implies that w.p. $\geq 1 - 2 \exp(n \log 9 - c \min(\varepsilon, \varepsilon^2)N)$, (a) $\lambda_{\max}(\frac{1}{N} \mathbf{W}'\mathbf{W})$ is smaller than $\lambda_{\max}(\frac{1}{N} \mathbb{E}[\mathbf{W}'\mathbf{W}]) + 4\varepsilon K^2$ and (b) $\lambda_{\min}(\frac{1}{N} \mathbf{W}'\mathbf{W})$ is larger than $\lambda_{\min}(\frac{1}{N} \mathbb{E}[\mathbf{W}'\mathbf{W}]) - 4\varepsilon K^2$.

Theorem 5.39 of [1] is a corollary of the second claim of Theorem 2.1 specialized to isotropic \mathbf{w}_j 's. In that case $\mathbb{E}[\mathbf{W}'\mathbf{W}] = N\mathbf{I}$ and thus, by using the remark above with ε appropriately set (use $\varepsilon = \frac{(\log 9)\sqrt{n}}{\sqrt{2c}\sqrt{N}} + \frac{t}{4K^2\sqrt{N}}$), we get Theorem 5.39 of [1].

A. An example application of Theorem 2.1

One example where both claims of the above result are useful but the result of Theorem 5.39 (or of Remark 5.40) of [1] does not suffice is in analyzing the initialization step of our recently proposed low-rank phase retrieval algorithm [2], [3]. In fact this is where we first used this generalization. The example given below is motivated by this application.

Consider n -length independent and identically distributed, standard Gaussian random vectors $\mathbf{a}_{i,k}$, i.e., $\mathbf{a}_{i,k} \stackrel{\text{iid}}{\sim} \mathcal{N}(0, \mathbf{I})$,

with $i = 1, 2, \dots, m$ and $k = 1, 2, \dots, q$; and n -length deterministic vectors \mathbf{x}_k , $k = 1, 2, \dots, q$. Assume that $q \leq n^2$. Consider bounding

$$b := \left| \frac{1}{m} \sum_{i=1}^m (\mathbf{a}_{i,k}' \mathbf{x}_k)^2 - \|\mathbf{x}_k\|^2 \right|$$

By applying item 1 of Theorem 2.1 with $N = m$ and $\mathbf{w}_j \equiv \mathbf{a}_{j,k}$, $b_k \leq \epsilon \|\mathbf{x}_k\|^2$ w.p. at least $1 - \exp(-c\epsilon^2 m)$. Such a bound holds for all $k = 1, 2, \dots, q$ w.p. at least $1 - q \exp(-c\epsilon^2 m)$. Thus, to ensure that this bound holds w.p. at least $1 - 1/\text{poly}(n)$, we need $m \geq \frac{c(\log n + \log q)}{\epsilon^2} = \frac{c \log n}{\epsilon^2}$ since $q \leq n^2$. Here $\text{poly}(n)$ means polynomial in n .

On the other hand, to apply [1, Theorem 5.39], we first need to upper bound the b_k 's as

$$\begin{aligned} b_k &= |\mathbf{x}_k' \left(\frac{1}{m} \sum_{i=1}^m \mathbf{a}_{i,k} \mathbf{a}_{i,k}' - \mathbf{I} \right) \mathbf{x}_k| \\ &\leq \|\mathbf{x}_k\|^2 \left\| \frac{1}{m} \sum_{i=1}^m \mathbf{a}_{i,k} \mathbf{a}_{i,k}' - \mathbf{I} \right\| \end{aligned}$$

With this, we can get the same bound as above on the b_k 's by applying [1, Theorem 5.39] with $t = \sqrt{m} 4K^2 \epsilon - C_K \sqrt{n}$ (or, equivalently, by applying item 2 of Theorem 2.1 above). But the bound would hold with probability lower bounded by $1 - \exp(n \log 9 - c\epsilon^2 m)$. For a given m , this is a much smaller probability. Said another way, one would need $m \geq \frac{cn}{\epsilon^2}$ for the probability to be high enough (at least $1 - 1/\text{poly}(n)$). This is a much larger lower bound on m than the earlier one.

To see an application of item 2 of Theorem 2.1, consider bounding

$$\tilde{b} := \left\| \frac{1}{mq} \sum_{k=1}^q \sum_{i=1}^m \mathbf{a}_{i,k} \mathbf{a}_{i,k}' f_k^2 - \frac{1}{q} \sum_{k=1}^q f_k^2 \right\|$$

where f_k 's are scalars. By conditioning on the f_k 's, we can apply item 2 of Theorem 2.1 on all the $N = mq$ vectors $(\mathbf{a}_{i,k} f_k)$ to conclude that $\tilde{b} \leq \epsilon_2 \max_k f_k^2$, w.p. at least $1 - 2 \exp(n \log 9 - c\epsilon_2^2 mq)$. Thus, the bound holds w.p. at least $1 - 1/\text{poly}(n)$ if $m \geq \frac{cn}{\epsilon_2^2}$.

Observe that the $\mathbf{a}_{i,k}$'s are isotropic independent sub-Gaussian vectors but $\mathbf{a}_{i,k} f_k$'s are not. In fact, $\mathbb{E}[\mathbf{a}_{i,k} \mathbf{a}_{i,k}' f_k^2] = f_k^2$ and hence the vectors $\mathbf{a}_{i,k} f_k$ also do not have the same second moment matrix for all k, i . As a result, we cannot apply Theorem 5.39 or Remark 5.40 of [1] to bound \tilde{b} if we want to average over all the mq vectors. To apply one of these, we first need to upper bound \tilde{b} as

$$\tilde{b} \leq \frac{1}{q} \sum_{k=1}^q \left\| \frac{1}{m} \sum_{i=1}^m \mathbf{a}_{i,k} \mathbf{a}_{i,k}' - \mathbf{I} \right\| f_k^2$$

Now using [1, Theorem 5.39], we get $\tilde{b} \leq \epsilon_2 \frac{1}{q} \sum_{k=1}^q f_k^2 \leq \epsilon_2 \max_k f_k^2$ w.p. at least $1 - 2 \exp(n \log 9 - c\epsilon_2^2 m)$. Observe that mq is replaced by m in the probability now. Thus, to get the probability to be high enough (at least $1 - \frac{1}{\text{poly}(n)}$) we will need $m \geq \frac{cn}{\epsilon_2^2}$ which is, once again, a much larger lower bound than what we got by applying item 2 of Theorem 2.1.

To understand the context, in [2], m is the sample complexity required for the initialization step of low-rank phase

retrieval to get an estimate of the low-rank matrix $\mathbf{X} := [\mathbf{x}_1, \mathbf{x}_2, \dots, \mathbf{x}_q]$ that is within a relative error $c\epsilon$ of the true \mathbf{X} with probability at least $1 - 1/\text{poly}(n)$. If we directly use the result from [1], we will need $m \geq cn/\epsilon^2$, where as if we use Theorem 2.1, we can get a lower bound that is smaller than cn (when q is large enough).

III. CONCLUSIONS

We proved a simple generalization of a result of Vershynin [1] for random matrices with independent, sub-Gaussian rows.

We should mention that the first claim of Theorem 2.1 can be further generalized for two different vectors \mathbf{z}_1 and \mathbf{z}_2 as follows: with the same probability, $|\mathbf{z}_1' \mathbf{D} \mathbf{z}_2| \leq 4\epsilon^2 K^2 (\|\mathbf{z}_1\|^2 + \|\mathbf{z}_2\|^2)$. This follows because, for two sub-Gaussian scalars, x, y , xy is sub-exponential with sub-exponential norm bounded by $c(\|x\|_{\psi_2}^2 + \|y\|_{\psi_2}^2)$ [4].

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APPENDIX

PROOF OF OUR RESULT

A. Preliminaries

As explained in [1], nets are a convenient means to discretize compact metric spaces. The following definition is [1, Definition 5.1] for the unit sphere. For an $\epsilon > 0$, a subset \mathcal{N}_ϵ of the unit sphere in \mathbb{R}^n is called an ϵ -net if, for every vector \mathbf{x} in the unit sphere, there exists a vector $\mathbf{y} \in \mathcal{N}_\epsilon$ such that $\|\mathbf{y} - \mathbf{x}\| \leq \epsilon$.

The covering number of the unit sphere in \mathbb{R}^n , is the minimal cardinality of an ϵ -net on it. In other words, it is the size of the smallest ϵ -net, \mathcal{N}_ϵ , on it.

Fact 1.1.

- 1) By Lemma 5.2 of [1], the covering number of the unit sphere in \mathbb{R}^n is upper bounded by $(1 + \frac{2}{\epsilon})^n$.
- 2) By Lemma 5.4 of [1], for a symmetric matrix, \mathbf{D} , $\|\mathbf{D}\| \leq \max_{\mathbf{x}: \|\mathbf{x}\|=1} \|\mathbf{x}' \mathbf{D} \mathbf{x}\| \leq \frac{1}{1-2\epsilon} \max_{\mathbf{x} \in \mathcal{N}_\epsilon} \|\mathbf{x}' \mathbf{D} \mathbf{x}\|$.

Thus, if $\epsilon = 1/4$, then $\|\mathbf{D}\| \leq 2 \max_{\mathbf{x} \in \mathcal{N}_{1/4}} \|\mathbf{x}' \mathbf{D} \mathbf{x}\|$ and the cardinality of the smallest such net is at most 9^n .

A r.v. x is sub-exponential if the following holds: there exists a constant K_e such that $\mathbb{E}[|x|^p]^{1/p} \leq K_e p$ for all integers $p \geq 1$; the smallest such K_e is referred to as the sub-exponential norm of x , denoted $\|x\|_{\varphi_1}$ [1, Section 5.2].

The following facts will be used in our proof.

Fact 1.2.

- 1) If \mathbf{x} is a sub-Gaussian random vector with sub-Gaussian norm K , then for any vector \mathbf{z} , (i) $\mathbf{x}' \mathbf{z}$ is sub-Gaussian with sub-Gaussian norm bounded by $K\|\mathbf{z}\|$; (ii) $(\mathbf{x}' \mathbf{z})^2$

is sub-exponential with sub-exponential norm bounded by $2K^2\|z\|^2$; and (iii) $(x'z)^2 - \mathbb{E}[(x'z)^2]$ is centered (zero-mean), sub-exponential with sub-exponential norm bounded by $4K^2\|z\|^2$. This follows from the definition of a sub-Gaussian random vector; Lemma 5.14 and Remark 5.18 of [1].

- 2) By [1, Corollary 5.17], if $x_i, i = 1, 2, \dots, N$, are a set of independent, centered, sub-exponential r.v.'s with sub-exponential norm bounded by K_e , then, for any $\varepsilon > 0$,

$$\Pr\left(\left|\sum_{i=1}^N x_i\right| > \varepsilon K_e N\right) \leq 2 \exp(-c \min(\varepsilon, \varepsilon^2)N).$$

- 3) If $x \sim \mathcal{N}(0, \bar{\Lambda})$ with $\bar{\Lambda}$ diagonal, then x is sub-Gaussian with $\|x\|_{\varphi_2} \leq c\sqrt{\lambda_{\max}}$.

B. Proof of Theorem 2.1

The proof strategy is similar to that of Theorem 5.39 of [1]. By Fact 1.2, item 1, for each j , the r.v.s $w_j'z$ are sub-Gaussian with sub-Gaussian norm bounded by $K\|z\|$; $(w_j'z)^2$ are sub-exponential with sub-exponential norm bounded by $2K^2\|z\|^2$; and $(w_j'z)^2 - \mathbb{E}[(w_j'z)^2] = z'(w_j w_j' - \mathbb{E}[w_j w_j'])z$ are centered sub-exponential with sub-exponential norm bounded by $4K^2\|z\|^2$. Also, for different j 's, these are clearly mutually independent. Thus, by applying Fact 1.2, item 2 (Corollary 5.17 of [1]) with $K_e = 4K^2\|z\|^2$ we get the first part.

To prove the second part, let $\mathcal{N}_{1/4}$ denote a $1/4$ -th net on the unit sphere in \mathbb{R}^n . Let $D := \frac{1}{N} \sum_{j=1}^N (w_j w_j' - \mathbb{E}[w_j w_j'])$. Then by Fact 1.1 (Lemma 5.4 of [1])

$$\|D\| \leq 2 \max_{z \in \mathcal{N}_{1/4}} |z' D z| \quad (1)$$

Since $\mathcal{N}_{1/4}$ is a finite set of vectors, all we need to do now is to bound $|z' D z|$ for a given vector z followed by applying the union bound to bound its maximum over all $z \in \mathcal{N}_{1/4}$. The former has already been done in the first part. By Fact 1.1 (Lemma 5.2 of [1]), the cardinality of $\mathcal{N}_{1/4}$ is at most 9^n . Thus, using the first part, $\Pr\left(\max_{z \in \mathcal{N}_{1/4}} |z' D z| \geq \frac{4\varepsilon K^2}{2}\right) \leq 9^n \cdot 2 \exp(-c \frac{\min(\varepsilon, \varepsilon^2)}{4} N) = 2 \exp(n \log 9 - c\varepsilon^2 N)$. By (1), we get the result.